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# A SURPRISE IN SUM RULES - MODULATING FACTORS

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## ABSTRACT

A generic physical situation is considered where  $\text{Im } \Pi$ , the imaginary part of polarization operator (generalized susceptibility), can be measured on a finite interval and the high frequency asymptotics (up to a few orders) of  $\Pi$  can be calculated theoretically. In such a case, it is desirable to derive an equivalent form of the Kramers-Kronig dispersion relation, the so-called sum rule, in which both the high-frequency part of  $\text{Im } \Pi$  in the dispersion integral and the high-order contribution to  $\Pi$  are suppressed. We provide a general framework for derivation of such sum rules, without any recourse to an infinite-order differential operator. We derive sum rules for a wide set of weight functions and show that any departure from the  $e^{-t}$  behaviour of the weight function in sum rules leads to modulating factors on the theoretical side of sum rules, providing its low frequency regularization. We argue that by including modulating factors one can extend the domain of validity of sum rules further to an intermediate region of frequencies and can account for “bumps” which were observed numerically on the phenomenological side of sum rules at “intermediate” frequencies.

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**1. Introduction.**- Kramers-Kronig dispersion relations for generalized susceptibility  $\Pi$ , such as [1]

$$\Pi(i\omega) = \frac{2}{\pi} \int_0^\infty \frac{\text{Im } \Pi(s)}{s^2 + \omega^2} s ds, \quad (1)$$

where  $\omega$  is a frequency, arise in many branches of physics. The reason is that dispersion relations are direct consequence of the analyticity of generalized susceptibility in the upper-half complex plane which, in turn, is believed to be the consequence of causality.

Generalized susceptibility  $\Pi$  determines the response of the system to a perturbation. Dispersion relation (1) relates the value of  $\Pi$  at imaginary frequency to the integral over its imaginary part,  $\text{Im } \Pi$ . In the generic physical situation to be considered here,  $\text{Im } \Pi$  is determined experimentally on a finite interval  $(0, \omega_0)$  via directly measurable quantities such as scattering cross sections. On the other hand, we shall suppose that a few orders of the asymptotic expansion of  $\Pi$  for  $\omega \gg 1$  can be calculated theoretically. An example is provided by quantum chromodynamics (QCD), where the generalized susceptibility is the polarization operator, defined via the current-current correlation function,

$$\Pi^{\mu\nu}(q) \equiv i \int e^{-iqx} \langle 0 | T(j^\mu(x) j^\nu(0)) | 0 \rangle d^4x = (g^{\mu\nu} q^2 - q^\mu q^\nu) \Pi(q^2), \quad (2)$$

and  $q$  is a four-momentum. The polarization operator satisfies the standard QCD dispersion relation [2]

$$-\frac{d}{dQ^2} \Pi(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im } \Pi(s)}{(s + Q^2)^2} ds, \quad (3)$$

where  $\Pi(Q^2) \equiv \Pi(-q^2)$ . The imaginary part,  $\text{Im } \Pi$ , is proportional to measurable cross sections such as that for  $e^-e^+$  (electron-positron) annihilation into hadrons and, for sufficiently small momenta, can be supplied from experiment. On contrary, one can get, for  $Q^2$  sufficiently large, the QCD representation for the polarization operator as (asymptotic) series,

$$Q^2 \left( -\frac{d}{dQ^2} \right) \Pi(Q^2) \sim a_0 + \sum_{n=2} \frac{a_n}{Q^{2n}}. \quad (4)$$

Here, the first term  $a_0$  corresponds to asymptotic freedom and the remaining coefficients  $a_n$  ( $n \geq 2$ ) parametrize power corrections, describing long-distance effects. They are given as the product of a short-distance term calculable within perturbation theory and a corresponding (quark, gluon, etc.) condensate which is unknown in theory [2]. Therefore, in the generic physical situation, it is desirable to find an equivalent form of (1), a *sum*

*rule*, in which both the high-energy part of  $\text{Im } \Pi$  and the high-order contribution to  $\Pi$  are suppressed, thereby reducing respective contributions of  $\text{Im } \Pi$  and  $\Pi$  for those frequencies and those orders for which our knowledge is incomplete.

The first step in this direction was performed by Shifman, Vainshtein, and Zakharov (SVZ hereafter) [2]. By applying the formal differential operator  $\hat{L}_M$ ,

$$\hat{L}_M \equiv \lim_{\substack{Q^2 \rightarrow \infty, n \rightarrow \infty \\ Q^2/n=M^2}} \frac{1}{(n-1)!} (Q^2)^n \left( -\frac{d}{dQ^2} \right)^{n-1}, \quad (5)$$

on both sides of dispersion relation (3) they derived the following equivalent form of the dispersion relation:

$$\frac{1}{\pi} \int_0^\infty e^{-s/Q^2} \text{Im } \Pi(s) ds \sim Q^2 \sum_{n=0} \frac{a_n}{n! Q^{2n}}, \quad (6)$$

called the SVZ *sum rule*. In (6), the contribution of  $\text{Im } \Pi$  to the dispersion integral is reduced for energies  $s \geq Q^2$  by an exponentially decreasing factor. On the right-hand side of (6), the contribution of high orders of perturbation theory for  $\Pi$  is suppressed due to the factor  $n!$  in the denominator. Therefore, using sum rule (6), theoretical predictions and experimental results can be tested in a more efficient way than using original dispersion relation (1). In particular, in the case of QCD, one hopes that the validity of sum rule (6) can be extended to intermediate energies and that the condensate values could be determined. A natural question arises whether one can provide a general framework for derivation of sum rules with a general weight function and specify their domain of validity.

**2. Results.**- To answer the above questions, we considered the class of weight functions of the form

$$\chi_{\alpha,\beta} = (1/\alpha) t^{(\beta/\alpha)-1} e^{-t^{1/\alpha}}, \quad (7)$$

where  $0 < \alpha \leq \beta$ , and

$$\chi_\gamma(t) = t^\gamma e^{-e^t} \quad (8)$$

with  $\gamma \geq 0$ . Our choice (7-8) of weight functions covers a wide range starting from monotonically decreasing to “Gaussian-like” (cf. Ref. [3]) weight functions.

Our main result (see the next section) is that a general sum rule with a weight function  $\chi(t)$  can be written as

$$\frac{1}{\pi} \int_0^\infty \chi^{(1)}(s/Q^2) \text{Im } \Pi(s) ds \sim Q^2 \sum_{n=0} (-1)^{(n-1)} \chi^{(n)}(\varepsilon_n/Q^2) \frac{a_n}{n! Q^{2n}}, \quad (9)$$

where  $\chi^{(n)}$  denotes the  $n$ th derivative of  $\chi(t)$ . Constants  $\varepsilon_n$  in (9),  $0 \leq \varepsilon_n \ll Q^2$ , are, in general, different for different  $n$ .

We shall show that physically reasonable sum rules arise only if  $\chi(t)$  together with all its derivatives is regular at the origin. Then, in the region  $\varepsilon_n \ll Q^2$ ,

$$\frac{1}{\pi} \int_0^\infty \chi^{(1)}(s/Q^2) \operatorname{Im} \Pi(s) ds \sim Q^2 \sum_{n=0} (-1)^{(n-1)} \chi_n \frac{a_n}{Q^{2n}}, \quad (10)$$

where the coefficients  $\chi_n$  are defined by

$$\chi(t) \equiv \sum_{n=0}^\infty \chi_n t^n. \quad (11)$$

Sum rule (10) includes all particular cases discussed previously in Refs. [2, 4]. Because weight functions are entire functions, the coefficients  $\chi_n$  as a function of  $n$  decreases to zero faster than any polynomial in the limit  $n \rightarrow \infty$ .

Sum rule (9) will be derived by employing theory of summability methods, without any recourse to infinite-order differential operator such as (5). This approach will also enable us to include into our consideration the important case of weight function  $\chi_\gamma$ . In contrast to  $e^{-t}$  where frequency is required to be imaginary, the weight function  $\chi_\gamma$  allows one to consider dispersion relation (1) for any frequency in the upper-half complex plane.

In Sec. 4, we shall show that unless  $\chi(t) = e^{-t}$ , one has to have  $\varepsilon_n > 0$ . Thus any departure from the  $e^{-t}$  behaviour of the weight function in the sum rules leads necessarily to  $Q^2$ -dependent factors  $\chi^{(n)}(\varepsilon_n/Q^2)$  on the theoretical side of the sum rules, which we shall call *modulating factors*. Sum rules as a function of parameters  $\alpha$ ,  $\beta$ , and  $\gamma$  [see Eqs. (7) and (8)] for a particular weight function are analyzed in Sec. 5. Domain of applicability of sum rules (6) and (9) and role of modulating factors are then discussed in Sec. 6.

**3. Derivation of general sum rules.**- In what follows, the term “summability method” will stand for a moment constant analytic (Borel-like) summability method [5, 6]. In the theory of summability methods, weight functions  $\chi_{\alpha,\beta}$  and  $\chi_\gamma$  are examples of the generating function  $\chi(t)$  which generates the moments  $\mu(n)$ ,

$$\mu(n) = \int_0^\infty \chi(t) t^n dt. \quad (12)$$

Any summability method is characterized by its generating function which, obviously, must decrease to zero faster than any power of  $t$  in the limit  $t \rightarrow \infty$ . The properties

of a summability method then depend on the properties of the corresponding moment function  $F(t)$ , determined in terms of moments  $\mu(n)$  [7],

$$F(t) \equiv \sum_{n=0}^{\infty} \frac{t^n}{\mu(n)}. \quad (13)$$

In the case of  $\chi_{\alpha,\beta}$ , the moments are  $\mu(n) = \Gamma(\alpha n + \beta)$  and the corresponding moment function is known to be the Mittag-Leffler function [5],

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \quad (14)$$

In the case of  $\chi_{\gamma}$ , the moments  $\mu(n)$  are growing roughly like  $(\ln n)^n$  and the corresponding moment function is analyzed in [6]. Throughout the paper, we shall speak of Borel summability if and only if  $\chi(t) = e^{-t}$ , in which case the moments are  $\mu(n) = n!$ . All other choices of  $\chi(t)$  will be referred to as the other summability methods.

For any weight function  $\chi(t)$  discussed here, the Cauchy kernel in dispersion integral (3) can be expressed as

$$\frac{1}{s + Q^2} = \int_0^{\infty} \chi(Q^2 t) F(-st) dt = \int_0^{\infty} \chi(st) F(-Q^2 t) dt, \quad (15)$$

where integrals in (15) are absolutely convergent [5, 6]. Representation (15) is the key relation which will allow us to represent the Cauchy kernel in dispersion integral (3) as a  $\chi$ -weighted integral and, at the same time, to find the corresponding modification of the right-hand side of (3).

In the next, we shall assume dispersion integral (3) to be absolutely convergent [8]. Otherwise, one can always consider, instead of (3), higher derivatives of the dispersion relation,

$$\frac{1}{n!} \left( -\frac{d}{dQ^2} \right)^n \Pi(Q^2) = \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im } \Pi}{(s + Q^2)^{n+1}} ds. \quad (16)$$

Relations of the type (16) do not bring any complications to our discussion, since absolute convergence of the integral in (15) allows us to differentiate within the sign of integration and represent any power of the Cauchy kernel as

$$\frac{1}{(s + Q^2)^n} \equiv \frac{1}{(n-1)!} \left( -\frac{d}{ds} \right)^{n-1} \frac{1}{s + Q^2} = \frac{1}{(n-1)!} \int_0^{\infty} F(-Q^2 t) \left( -\frac{d}{ds} \right)^{n-1} \chi(st) dt. \quad (17)$$

Again, for any  $n$  the integral in (17) converges absolutely [5, 6]. Then, by using relation (16) for  $n = 1$ , dispersion relation (3) can be written in the form

$$-\frac{1}{\pi} \int_0^\infty \frac{1}{M^2} F(-Q^2/M^2) \left( \int_0^\infty \chi^{(1)}(s/M^2) \operatorname{Im} \Pi(s) ds \right) d\frac{1}{M^2} = \sum_{n=0} \frac{a_n}{Q^{2(n+1)}}. \quad (18)$$

Now, the key problem is to find the function  $G(1/M^2)$  such that the right-hand side of (18) can be represented as an integral of the form

$$\sum_{n=0} \frac{a_n}{Q^{2(n+1)}} \equiv \int_0^\infty \frac{1}{M^2} F(-Q^2/M^2) G(1/M^2) d\frac{1}{M^2}. \quad (19)$$

In order to find  $G(1/M^2)$ , one again uses representation (15) and (17) of the Cauchy kernel and its powers. Since  $Q^2$  is assumed to be sufficiently large, one introduces only small error if  $1/Q^{2n}$  is approximated by  $1/(Q^2 + \varepsilon)^n$ , where  $0 < \varepsilon \ll Q^2$ . Then, according to (17), general power  $1/Q^{2n}$  can be approximated as

$$\frac{1}{Q^{2n}} \sim \frac{1}{(Q^2 + \varepsilon)^n} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty F(-Q^2 t) \chi^{(n-1)}(\varepsilon t) t^{n-1} dt, \quad (20)$$

which introduces an infinitesimally small correction to higher orders. In what follows, we shall first approximate the right-hand side of (18) according to (20) and we shall discuss the limit  $\varepsilon \rightarrow 0$  afterwards. Now, in the  $\varepsilon$ -approximation,

$$G(1/M^2) = M^2 \sum_{n=0} (-1)^n \chi^{(n)}(\varepsilon/M^2) \frac{a_n}{n! M^{2n}}, \quad (21)$$

and comparison with Eqs. (18) and (19) leads immediately to general sum rule (9).

**4. Modulating factors.**- A natural question arises what is the role of these yet unspecified parameters  $\varepsilon_n$ . Can one get rid of them?

In the case of the Borel method, the generating function  $\chi(t)$  and the moment function  $F(-t)$  are identical,

$$\chi(t) \equiv F(-t). \quad (22)$$

We shall refer to this property as “self-duality”. It means that both  $\chi(t)$  and  $F(-t)$  are exponentially decreasing and integrals in (18), (19), and (20) are absolutely convergent Laplace integrals. In this case, obviously,  $\varepsilon$  can be sent to zero in (20),

$$G(1/M^2) = M^2 \sum_{n=0} \frac{a_n}{n! M^{2n}}, \quad (23)$$

and one recovers the SVZ sum rule (6).

For all other moment constant summability methods discussed here, the “self-duality” (22) is lost,

$$\chi(t) \neq F(-t). \quad (24)$$

Even more significant than the lack of “self-duality” property (22) for the other summability methods is that, in the case of the Borel method,  $F(-t)$  ( $= e^{-t}$ ) approaches zero faster than the inverse of any polynomial in the limit  $t \rightarrow \infty$ , while such inverse polynomial decrease is characteristic for all other summability methods. Indeed, for all other summability methods, the asymptotic behaviour of  $F(-t)$  in the limit  $t \rightarrow \infty$  is characterized by an abrupt switch off from the *exponential* decrease  $e^{-t}$  to a *universal polynomial* decrease [9],

$$F(-t) \sim \mathcal{O}(t^{-1}) \quad (t \rightarrow \infty). \quad (25)$$

For example, in the case of  $\chi_{\alpha,\beta}(t)$  the moment function is  $E_{\alpha,\beta}(t)$  [see (14)], having the asymptotic behaviour [5]

$$E_{\alpha,\beta}(-t) \sim \sum_{n=1}^{\infty} \frac{(-t)^n}{\Gamma(\beta - \alpha n)} \quad (t \rightarrow \infty). \quad (26)$$

Unless both  $\beta$  and  $\alpha$  are integers and  $\beta \leq \alpha$ , polynomial terms in the asymptotic expansion (26) of  $E_{\alpha,\beta}(-t)$  are always present. Indeed, Eq. (26) implies that  $E_{\alpha,\beta}(-t)$  is exponentially decreasing if and only if for all  $n$ ,  $\beta + \alpha n$  is a pole of the Euler gamma function  $\Gamma(z)$ . For  $\chi_{\gamma}(t)$  see [6].

The loss of the exponential decrease of  $F(-t)$  for all other summability methods implies that the limit  $\varepsilon \rightarrow 0$  cannot be taken. Because of the universal behaviour (25) of  $F(-t)$  at infinity,

$$\int_0^\infty F(-Q^2/M^2) \left( \frac{1}{M^2} \right)^n d\frac{1}{M^2} \quad (27)$$

diverges for any  $n \geq 0$  and  $G(1/M^2)$  *cannot* be pure polynomial. It must contain modulating factors which ensure convergence of (19) for small  $M^2$ . Therefore, in general case, the sum rule corresponding to the weight (generating) function  $\chi(t)$  must be given by relation (9) with  $0 < \varepsilon_n \ll Q^2$ .

**5. Analysis of sum rules for different weight functions.**- Let us first consider  $\chi_{\alpha,\beta}$  and denote  $p \equiv \beta/\alpha - 1$  and  $q \equiv 1/\alpha$ . Then, in order to have meaningful sum rules,  $p$  and

$q$  must be respectively nonnegative and positive integers. Otherwise,  $t^n \chi_{\alpha,\beta}^{(n)}(t) \sim \chi_{\alpha,\beta}(t)$  in the limit  $t \rightarrow 0$ , resulting in the fact that all orders in (4) will behave as  $\mathcal{O}(1)$  in the limit  $Q^2 \rightarrow \infty$  on the theoretical side of sum rule (9) and thus, they will merge to a single order and become *equivalent*.

If  $p$  is a nonnegative integer and  $q$  is a positive integer, then  $\chi_{\alpha,\beta}$ , including all its derivatives, is regular at the origin. Therefore, for  $\varepsilon_n \ll Q^2$ , modulating factors  $\chi_{\alpha,\beta}^{(n)}(\varepsilon_n/Q^2)$  in (9) can be approximated by their values at the origin and one finds relation (10) which, upon appropriate choice of the weight function, gives the SVZ sum rule (6) and its generalizations discussed in [4].

To discuss particular cases, one uses

$$\chi^{(n)}(0) = n! \chi_n = (-1)^l q \frac{(p+lq)!}{l!} \delta_{n,p+lq} = (-1)^l q \frac{n!}{l!} \delta_{n,p+lq}. \quad (28)$$

Therefore, unless  $n = p + lq$ , where  $l$  is a nonnegative integer,  $\chi_{\alpha,\beta}^{(n)}(0) = 0$ . Relation (28) implies that the sum rule with such weight function is (i) *selective*: it only selects the orders  $n = p + lq$  of the asymptotic expansion (4), (ii) the contribution of the  $(p + lq)$ th order in (4) is then reduced by the factor  $1/l!$  on the theoretical side of the sum rule. For example, in the case of  $\chi(t) = 2e^{-t^2}$  only even terms of the asymptotic expansion (4) enter the sum rule (10). Thus, such selective sum rules can be particularly useful in the case when some symmetries of the problem ensure that only  $(p + lq)$ th orders of the asymptotic expansion (4) are nonzero.

In the case of  $\chi_\gamma$ , similar argument shows that it is useful to have  $\gamma \neq 0$  only if the first  $\gamma$  orders of the asymptotic expansion (4) are identically zero. Otherwise, for  $e^{-e^t}$ , one has

$$\frac{d^n}{dt^n} \exp(-e^t) \equiv n! \bar{\chi}_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} k^n. \quad (29)$$

Estimates, using Euler-Maclaurin sum formula [10], give  $|\bar{\chi}_n| \sim (\ln n)^n/n!$  in the limit  $n \rightarrow \infty$ .

**6. Domain of validity of sum rules and physical significance of modulating factors.-** Sum rules (6) and (9) are derived for  $Q^2$  sufficiently large. However, the meaning of “sufficiently large” must be decided separately in each particular case. Since, eventually, one wants to make some practical use of sum rules, one is interested in some intermediate region of energies (or frequencies). The best possible situation occurs if there is some

nonempty intersection ("fiducial region") of respective intervals on which  $\text{Im}\Pi$  and  $\Pi$  are known. Note that if the intersection is empty, one may gain by repeating all our calculations with  $\chi(t)$  replaced by  $\chi_A(t) = \chi(At)$ , by introducing a scaling variable  $A$ . This scaling leads to  $F(t)$  replaced by  $F_A(t) = F(At)$  and  $n!$  factor in (21) replaced by  $n!/A^n$ . The corresponding sum rule is then

$$\frac{1}{\pi} \int_0^\infty \chi^{(1)}(sA/Q^2) \Pi(s) ds \sim Q^2 \sum_{n=0} (-1)^{(n-1)} \chi^{(n)}(\varepsilon_n A/Q^2) a_n \frac{A^n}{n!Q^{2n}}. \quad (30)$$

For  $A > 1$ , one gains on the experimental side of sum rule: the interval on which  $\text{Im}\Pi$  must be accurately determined is scaled down from interval  $(0, Q^2)$  onto interval  $(0, Q^2/A)$ . However, one lost on the theoretical side of the sum rule: (i) damping of higher order terms is slower, (ii) since the factor  $A^n/n!$  has its maximum at  $n = A$ , terms in the expansion (4) at the order  $n \approx A$  enter the sum rule with relatively the highest weight. Therefore, although the scale transformation is worth considering, it is not at all obvious whether one can extend the validity of sum rules to the intermediate region.

We argue that modulating factors play important role in the extension of validity of sum rules to the intermediate region. The argument is as follows: by comparing the behaviour of both sides in (6) in the limit  $Q^2 \rightarrow 0$  one finds that the left-hand side goes to zero while the right-hand side diverges. On the other hand, if modulating factors are present then, in contrast to (6), both sides of (9) tend to zero in the limit  $Q^2 \rightarrow 0$ . Modulating factors can be viewed as the low energy regularization of sum rules. The mismatch between the vanishing rates of the two sides of (9) in this limit (the left-hand side tends to zero polynomially while the right-hand side tends to zero faster than any power of  $Q^2$ ) is not too important: nobody expects sum rule (9) to be valid for  $Q^2$  too small. Nevertheless, the very fact that, in the presence of modulating factors, both sides of (9) tend to zero in the limit  $Q^2 \rightarrow 0$ , represents substantial improvement over the SVZ sum rule.

Another argument to support our claim that modulating factors can extend the validity of sum rules to the intermediate region is the appearance of "bumps" observed numerically on the phenomenological side of sum rules in [4] at  $Q^2 \approx 0.5\text{GeV}^2$  (see Fig. 1). In the absence of modulating factors, it was impossible to match the fits of both sides of sum rule (10) in the intermediate energy region and a conclusion was drawn that the Borel summability is optimal and that by any departure from the Borel summability sum rules

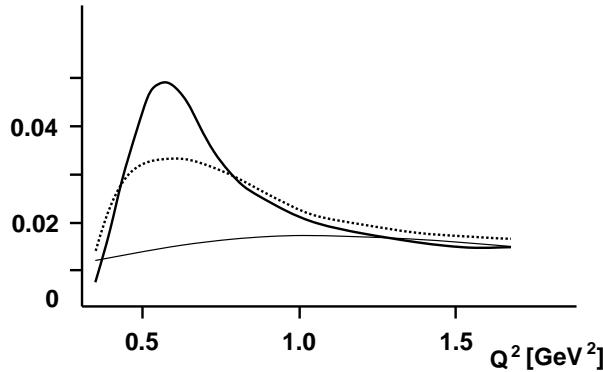


Figure 1: Formation of a “bump” on the phenomenological side of sum rule for  $q = 1, 2$ , and  $3$ . As  $q$  increases, the bump becomes more and more pronounced.

get worse [4]. However, if modulating factors are included, then, since  $G(1/Q^2) \rightarrow 0$  in the limit  $Q^2 \rightarrow 0$ ,  $G(1/Q^2)$  will develop a maximum at the point where  $\chi^{(n-1)}(\varepsilon_n/Q^2)$  overcomes polynomial increase of  $1/Q^{2n}$  (in the limit  $Q^2 \rightarrow 0$ ). This turning point then provides a way of determining modulating constant  $\varepsilon_n$  phenomenologically.

**7. Discussion and conclusions.-** We have provided a general derivation of sum rules within the framework of summability methods, without any recourse to a formal differential operator (5) (cf. [2, 4]).

We have shown that unless symmetries of a physical model cause some particular orders of the asymptotic expansion (4) of the polarization operator  $\Pi$  to vanish, the only reasonable choices of weight functions, from the set (7) and (8) we have considered, are  $e^{-t}$  and  $e^{-e^t}$ . The use of  $e^{-e^t}$  instead of  $e^{-t}$  in sum rules means also a significant *qualitative change*: due to very special properties of the summability method based on the weight function  $e^{-e^t}$  [11], one is not bound to consider dispersion relation (1) for only purely imaginary frequencies as in the case of  $e^{-t}$  but, instead, one can consider dispersion relation and derive sum rules for any frequency in the upper-half complex plane. This can have useful applications in other physical models. Moreover, a sum rule with  $e^{-e^t}$  has two advantages over sum rules with  $e^{-t}$ :

- the high-energy part of the integral is cut-off more effectively;
- the onset of the cut-off in the integral (9) starts sooner, already at  $s \sim Q^2 \ln 2$  and not at  $s \sim Q^2$  as in the latter case.

All the above advantages may compensate for a slower,  $(\ln n)^n/n!$  damping [cf. Eq. 29)] of higher orders of expansion (4) in comparison to  $1/n!$  damping in the case of  $e^{-t}$ .

Further, we have proved that if a general weight function is used, the sum rule must contain modulating factors. The latter provide the low frequency regularization of sum rules and could extend their validity up to the intermediate region of energies. The very existence of the modulating factors in a sum rule with weight function different from  $e^{-t}$  results from the universal  $1/t$  decay [see Eq. (25)] of the moment function  $F(-t)$ . It is amazing to note that a modulating factor can be also present in the case of the Borel summability method, and may lead to essential improvement of the fit. One cannot rule out its presence unless one has absolute control over numerical values of expansion constants in the asymptotic behaviour (4) of  $\Pi$ . If not, one can always find a positive constant  $\varepsilon$  such that replacement of  $1/Q^2$  by  $1/(\varepsilon + Q^2)$  makes a change in the expansion (4) which is within the error due to our incomplete knowledge of the expansion constants  $a_n$ . Then, the use of (20) inevitably leads to a modulating factor. The only exception of the Borel summability method is that, in principle, the modulating constants  $\varepsilon_n$  can be sent to zero and the limit exists [giving the SVZ sum rule (6)], while for all other summability methods, the modulating constants *must* be nonzero. The modulating factors were not seen using the formal infinite-order differential operator  $\hat{L}_M$  involving two limiting procedures [cf. Eq. 5)], because the latter set the argument of the modulating factors to the origin rendering them  $Q^2$  independent.

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## References

- [1] L. D. Landau and E. M. Lifshitz, em Statistical Physics Part 1, 3rd ed., (Pergamon Press, London, 1989), pp. 382-384.
- [2] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979).

- [3] R. A. Bertlmann, G. Launer, E. de Rafael, Nucl. Phys. B **250**, 61 (1985).
- [4] J. Fischer and P. Kolář, Z. Phys. C **34**, 375 (1987).
- [5] G. Hardy, *Divergent series* (Oxford: Oxford Univ. Press, 1949)
- [6] A. Moroz, Czech. J. Phys. B **40**, 705 (1990); Commun. Math. Phys. **133**, 369 (1990); PhD thesis, Prague Institute of Physics, 1991 (compuscript without figures available in the hep-th network as hep-th/9206074); Czech. J. Phys. B **42**, 753 (1992). Only the case  $\chi_0$  is considered there. However, the generalization to  $\chi_\gamma$  is straightforward.
- [7] Functions  $\chi(t)$  and  $F(t)$  play important roles in the theory of summability methods. If  $f(z)$  has an (either asymptotic or the Taylor) expansion  $f(z) = \sum_n a_n z^n$ , then  $f(z) = \int_0^\infty \chi(zt) T(f)(t) dt$ , where locally, within its radius convergence,  $T(f)(t) = \sum_n a_n t^n / \mu(n)$ . Outside its radius convergence, analytic continuation  $T(f)(t)$  is given by  $T(f)(t) = \oint_C F(t/z) f(z) dz$ , where  $C$  is contour specified by the condition that  $\operatorname{Re} \ln F(t/z) = \operatorname{const}$ . See [6] for details.
- [8] We remind the reader that  $\operatorname{Im} \Pi$  has a definite sign,  $\operatorname{Im} \Pi \leq 0$ , and the convergence of the dispersion integral in (3) is equivalent to its absolute convergence.
- [9] Except for cases where  $\beta = \alpha$ , in which case, provided that  $\alpha$  is not an integer, the leading term in the asymptotic expansion of  $F(-t)$  is of order  $t^{-2}$ . If  $\alpha$  is an integer and  $\alpha \geq 2$ , weight function  $\chi(t)$  decays as  $e^{-\sqrt{t}}$  or slower which is not interesting for our purposes.
- [10] M. V. Fedoriuk, *Asymptotic estimates: Integrals and series* (in Russian) (Nauka, Moscow, 1987); M. A. Evgrafov, *Asymptotic estimates and entire functions* (in Russian) (Nauka, Moscow, 1979).
- [11] This method allows us to represent the Cauchy kernel in (3) as the integral (15) for all complex  $Q^2 \notin (-\infty, 0]$  [6]. The method also enables us to sum asymptotic series even in the case where the sum is known to be analytic only in a horn-shaped region with zero opening angle [6]. Recently, this method has been applied to calculate critical coefficients for the nonlinear  $\sigma$ -model on the symmetric space  $\operatorname{Sp}(N)/[\operatorname{Sp}(N-p) \otimes \operatorname{Sp}(p)]$  in the replica ( $N, p \rightarrow 0$ ) limit [see A. Moroz, J. Phys. A **29**, 289 (1996)].